7.4 (continued)

multiplication: if $\mathcal{L}\{h(t)\} = F(s)\,G(s)$, what is $h(t)$?

it is NOT $f(t)g(t)$ in general

$$\mathcal{L}\{f(t)g(t)\} \neq F(s)\,G(s)$$

it is actually, the convolution integral

$$\boxed{\begin{gathered} f(t) * g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \\ \mathcal{L}\{f(t) * g(t)\} = F(s)\,G(s) \end{gathered}}$$

this integral has some important meanings we will see in 7.6

for now, it's an alternative to perform inverse Laplace transform for example, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

option 1: partial fraction

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

option 2: integral property

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

inverse

option 3: convolution

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F(s)}\ \underbrace{\frac{1}{s^2+1}}_{G(s)}\right\}$$

$f(t) = 1$

$g(t) = \sin(t)$

$$= \int_0^t \underbrace{1}_{f(\tau)} \cdot \underbrace{\sin(t-\tau)}_{g(t-\tau)}\,d\tau = \int_0^t \underbrace{1}_{f(t-\tau)} \cdot \underbrace{\sin(\tau)}_{g(\tau)}\,d\tau = -\cos(\tau)\Big|_0^t$$

$$= -\cos(t) + 1$$

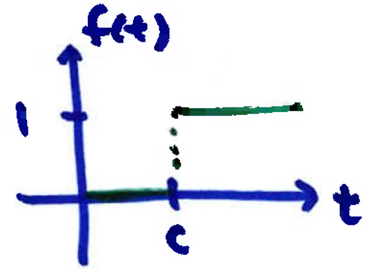
7.5 Discontinuous Input Functions

$$ay'' + by' + cy = f(t)$$

discontinuous is ok as long as it's piecewise continuous

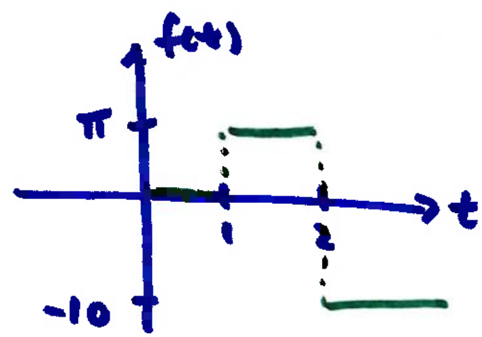
one tool: unit step function

$$u_c(t) = u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{else} \end{cases}$$



we can write many functions in terms of $u_c(t)$

$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ \pi & 1 \leq t < 2 \\ -10 & 2 \leq t < \infty \end{cases}$$



reset to 0

$$= 0 + u_1(t)(\pi) + u_2(t)(-\pi - 10)$$

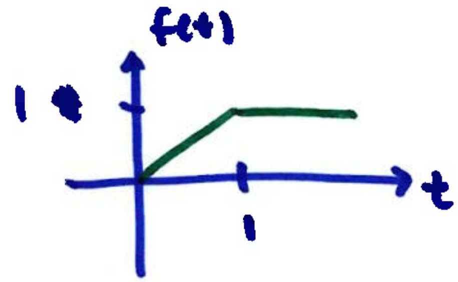
switch on at t = 1

this happens at t = 1

what we want

try this one:

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$



reset to 0

desired level

$$= t + u_1(t)(-t+1)$$

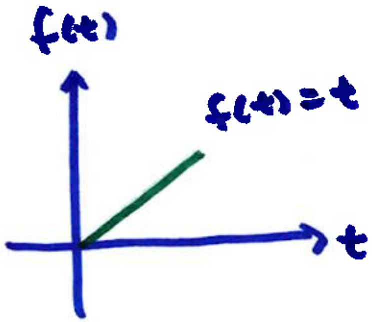
$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\, e^{-st}\, dt \qquad u_c = \begin{cases} 0 & t < c \\ 1 & t \ge c \end{cases}$$

$$= \int_0^c 0 \cdot e^{-st}\, dt + \int_c^\infty 1 \cdot e^{-st}\, dt$$

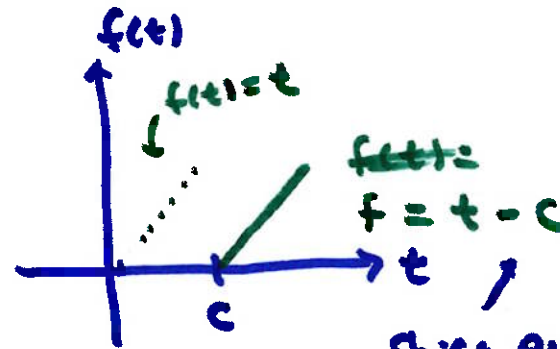
$$= -\frac{1}{s} e^{-st} \Big|_{t=c}^{t=\infty} = \frac{1}{s} e^{-sc}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}}$$

more general: $\mathcal{L}\{\underbrace{u_c(t) f(t-c)}_{\text{delayed activation of } f(t) \text{ by } c}\}$



same shape of t at later time $\longrightarrow$



shift RIGHT by c

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty u_c(t) f(t-c) e^{-st}\,dt \qquad u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$$= \int_c^\infty f(t-c) e^{-st}\,dt$$

let $\tau = t - c$ $\quad t = \tau + c$

$d\tau = dt$

$$= \int_0^\infty f(\tau) e^{-s(\tau+c)}\,d\tau$$

$$= e^{-sc} \underbrace{\int_0^\infty f(\tau) e^{-s\tau}\,d\tau}_{\mathcal{L}\{f(\tau)\}}$$

τ is variable, so rename

$\mathcal{L}\{f(\tau)\} = \mathcal{L}\{f(t)\} = F(s)$

$$\boxed{\mathcal{L}\{u_c(t)\,f(\underline{t-c})\} = e^{-cs}\underbrace{\mathcal{L}\{f(\underline{t})\}}}$$

transform $f(t-c)$ shifted back to origin : change t to $t+c$ (~~R~~LEFT shift by c)

$$\mathcal{L}\{u_1(t)\underbrace{(1-t)}_{f(t-1)}\} = e^{-s}\mathcal{L}\{\underbrace{1-(t+1)}_{f(t)}\}$$

$$= e^{-s}\mathcal{L}\{-t\} = \frac{-e^{-s}}{s^2}$$